Quantum inflaton dynamics

Dongsu Bak,[1,*] Sang Pyo Kim,[2,†] Sung Ku Kim,[3,‡] Kwang-Sup Soh,[4,§] and Jae Hyung Yee[5,‖]
[1]*Department of Physics, University of Seoul, Seoul 130-743, Korea*
[2]*Department of Physics, Kunsan National University, Kunsan 573-701, Korea*
[3]*Department of Physics, Ewha Womans University, Seoul 120-750, Korea*
[4]*Department of Physics Education, Seoul National University, Seoul 151-742, Korea*
[5]*Department of Physics, Yonsei University, Seoul 120-749, Korea*
(Received 7 April 1998; published 25 November 1998)

We show that the quantum dynamics of a real scalar field for a large class of potentials in the symmetric Gaussian state, where nonperturbative quantum contributions are taken into account, can be described equivalently by a two-dimensional nonlinear dynamical system with a definite angular momentum [U(1) charge of a complex theory]. A proposal is put forward that the symmetric Gaussian state with a nearly minimal uncertainty and a large quantum fluctuation, as an initial condition, naturally explains most of the essential features of the early stage of the inflationary universe. [S0556-2821(98)02024-4]

PACS number(s): 98.80.Cq, 04.62.+v

The standard cosmological model combined with particle physics at high energy (temperature) is plagued with the monopole, horizon, flatness problems, and so on. A major breakthrough to solve most of these problems was found in the inflation paradigm invented by Guth [1]. (For a review and references, see [2].) His simple but attractive idea, the so-called old inflation scenario, is that an inflaton (homogeneous scalar field) undergoes a first order phase transition from a symmetric vacuum as the Universe expands and thereby the temperature drops, and the energy density of the inflaton captured in the false vacuum drives an exponential expansion of the de Sitter phase. However, this scenario has a major defect — the graceful exit problem [3]. The new inflation scenario [4] based on a second order phase transition overcomes the graceful exit problem, but raises another problem involving a fine-tuning of the coupling constants. It is also possible to solve the graceful exit problem in the context of Jordan-Brans-Dicke theory [5]. The chaotic inflation scenario introduced by Linde [6] does not make any use of phase transitions but rather investigates various initial distributions of the inflaton that lead to inflation. Furthermore, this scenario is model independent in the sense that one can obtain a significant inflation for a large class of potentials.

Most of these scenarios are based on the classical gravity of the Friedmann equation and the scalar field equation on the Friedmann-Robertson-Walker (FRW) universe, assuming its validity even at the very early Universe. However, quantum effects of matter fields such as initial quantum conditions and quantum fluctuations are expected to play a significant role in this regime, though quantum gravity effects are still negligible. The proper description of a cosmological model should be in terms of the semiclassical gravity of the semiclassical Friedmann equation with quantized matter fields as far as inflation is concerned. There are a few works in which the quantum properties of the inflaton were investigated and used in inflation scenarios. In the new inflation scenario the quantum effects of the inflaton were partially taken into account by using a one-loop effective potential and an initial thermal condition in Ref. [7] and in the stochastic inflation scenario the inflaton was studied quantum mechanically by dealing with the phase-space quantum distribution function or the probability distribution [8].

In this paper, we pursue the quantum dynamics of a real-valued inflaton (homogeneous real scalar field) for a large class of potentials in the FRW universe and put forward a proposal for the initial quantum condition for the inflationary universe. In the Schrödinger picture, the scalar inflaton theory is approximately but quite accurately described by a symmetric Gaussian state for an extremal configuration of a bosonic condensate of scalar particles. The symmetric Gaussian state can be explicitly found by using the nonperturbative method of Ref. [9] or a direct Gaussian wave function as in Refs. [10,11]. The resulting equations are two-dimensional coupled nonlinear equations where the dynamical variables are the dispersion of the inflaton and the scale factor of the FRW universe. This semiclassical quantum description of gravity shows many features similar to classical gravity except that nonperturbative quantum contributions are taken into account, affect the coupling constants, a conserved angular momentum appears from the quantization condition, and so on. Moreover, the semiclassical gravity approach provides a more direct and clear description for the initial quantum conditions than the phase-space quantum distribution function or probability distribution [8]. In the end, within the framework of semiclassical gravity, we propose that the symmetric Gaussian state with a large fluctuation (dispersion) and a minimal uncertainty is the initial quantum condition for the inflationary universe.

We shall begin with a spatially flat FRW metric[1]

*Electronic address: dsbak@mach.scu.ac.kr
†Electronic address: sangkim@knusun1.kunsan.ac.kr
‡Electronic address: skkim@theory.ewha.ac.kr
§Electronic address: kssoh@phya.snu.ac.kr
‖Electronic address: jhyee@phya.yonsei.ac.kr

[1]We shall use the unit system $c=\hbar=1$ and $m_P^2=1/G$ and follow the notation of Ref. [6], in which the Planck length and time are $l_P=t_P=1/m_P$, ϕ and a have the dimension m_P and $1/m_P$, respectively, and λ_{2n} is dimensionless.

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{1}$$

and consider a real homogeneous and isotropic inflaton described by the Hamiltonian

$$H(t) = \frac{p^2}{2a^3(t)} + a^3(t)\left[\frac{m^2}{2}\phi^2 + \frac{m_P^4\lambda_{2n}}{(2n)}\left(\frac{\phi}{m_P}\right)^{2n}\right], \tag{2}$$

where $p = a^3\dot{\phi}$ is the momentum conjugate to the real inflaton field. The classical equation of motion for the inflaton field reads

$$\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\phi} + m^2\phi + m_P^2\lambda_{2n}\left(\frac{\phi}{m_P}\right)^{2n-2}\phi = 0, \tag{3}$$

while the scale factor $a(t)$ is governed by the Friedmann equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi}{3a^3 m_P^2}H(t). \tag{4}$$

In this classical context, it is known that the inflation may indeed occur but the unnatural fine-tuning of the initial data is, more or less, needed as will be seen. But before dealing with classical gravity, we shall first analyze quantum theory and obtain the classical theory as its limit.

Upon quantization of the real inflaton, the Schrödinger equation for a quantum field,

$$i\frac{\partial}{\partial t}\Psi(\phi,t) = \hat{H}(t)\Psi(\phi,t), \tag{5}$$

determines the time evolution of the inflaton, whereas the Friedmann equation, at its semiclassical level,

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi}{3a^3 m_P^2}\langle\hat{H}\rangle, \tag{6}$$

describes the evolution of the Universe.

There is a rather standard method to find an approximate Gaussian state that extremizes the energy (see, e.g., Refs. [9–11]). Although one may use a more generic Gaussian wave function, we shall use a symmetric Gaussian state [10]

$$\Psi(\phi,t) = \frac{1}{[2\pi\chi^2(t)]^{1/4}}\exp\left[-\left(\frac{1}{4\chi^2(t)} - i\frac{\pi_\chi(t)}{2\chi(t)}\right)\phi^2\right], \tag{7}$$

as our trial state. The real functions χ and π_χ are time-dependent parameters, whose time dependence will be determined below by the Dirac action principle. We are going to extremize the effective action

$$I_{\rm eff} = \int dt\langle\Psi|[i\partial_t - \hat{\mathbf{H}}(t)]|\Psi\rangle, \tag{8}$$

which is the Dirac action except for its wave function being now limited by the Gaussian form in Eq. (7). By a straightforward computation, one finds that the effective action is given by

$$I_{\rm eff} = \int dt[\pi_\chi\dot{\chi} - H_{\rm eff}(\pi_\chi,\chi)], \tag{9}$$

where

$$H_{\rm eff}(\pi_\chi,\chi) = \frac{\pi_\chi^2}{2a^3} + a^3 V_{\rm eff}(\chi), \tag{10}$$

with the effective potential

$$V_{\rm eff}(\chi) = \frac{1}{8a^6\chi^2} + \frac{m^2}{2}\chi^2 + \frac{m_P^4\lambda_{2n}^Q}{(2n)}\left(\frac{\chi}{m_P}\right)^{2n}, \tag{11}$$

where $\lambda_{2n}^Q = [(2n)!/2^n n!]\lambda_{2n}$ is an effective coupling constant. Upon extremization of the effective action (9), one finds that our system is described by Hamilton's equations

$$a^3\dot{\chi} = \pi_\chi,$$

$$\dot{\pi}_\chi = -a^3\frac{\partial}{\partial\chi}V_{\rm eff}(\chi) = \frac{1}{4a^3\chi^3} - a^3\left[m^2\chi + m_P^3\lambda_{2n}^Q\left(\frac{\chi}{m_P}\right)^{2n-1}\right]. \tag{12}$$

The semiclassical Friedmann equation becomes

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi}{3a^3 m_P^2}H_{\rm eff}. \tag{13}$$

One can also understand the physics of the above system from a different point of view using the method of Ref. [9], which is based on the technique of solving the Schrödinger equation for the time-dependent Hamiltonian system [12]. One introduces the annihilation operator of a Fock space [13] redefined as a dimensionless quantity:

$$\hat{A} = \frac{\varphi^*(t)}{m_P}\hat{p} - m_P a^3(t)\dot{\varphi}^*(t)\hat{\phi}, \quad \hat{A}^\dagger = \text{H.c.}, \tag{14}$$

such that

$$[\hat{A},\hat{A}^\dagger] = ia^3(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) = 1, \tag{15}$$

and $\varphi(t)$ in Eq. (14) is a complex scalar variable with the same dimension as the classical field ϕ. One then expands the Hamiltonian in terms of $\hat{A}$ and $\hat{A}^\dagger$, and truncates it up to the quadratic terms $\hat{H}_{(2)}$ for an approximation. The requirement that $\hat{A}$ and $\hat{A}^\dagger$ should be solutions of the Liouville-Neumann equation

$$i\frac{\partial\hat{A}}{\partial t} + [\hat{A},\hat{H}_{(2)}] = 0 \tag{16}$$

leads to the equation of motion for the φ field

$$\ddot{\varphi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\varphi} + m^2\varphi + m_P^2\lambda_{2n}^Q\left(\frac{\varphi^*\varphi}{m_P^2}\right)^{n-1}\varphi = 0. \tag{17}$$

If one sets $\varphi = \chi e^{-i\theta}$, Eq. (17) reduces to Eq. (12) with the constraint (15) expressed as

$$Q \equiv 2a^3\chi^2\dot{\theta} = 1. \tag{18}$$

We have thus found that the quantum dynamics of the real inflaton theory is equivalently described by the two-dimensional nonlinear dynamical system. The expectation value of the Hamiltonian gives rise to the effective Hamiltonian

$$H_{\rm eff}(t) = \langle \hat{H}(t)\rangle = a^3\left\{\frac{1}{2}\dot{\varphi}^*\dot{\varphi} + \frac{m^2}{2}\varphi^*\varphi + \frac{m_P^4\lambda_{2n}^Q}{(2n)}\left(\frac{\varphi^*\varphi}{m_P^2}\right)^n\right\}. \tag{19}$$

It should be noted that this complex system has a U(1) symmetry under the transformation $\varphi \to e^{i\alpha}\varphi$, and that the commutation relation $[\hat{A},\hat{A}^\dagger] = 1$ indeed determines the charge to be unity. In this framework one can easily obtain higher order corrections to the Gaussian state (7).[2]

A few comments are in order. First, we note that the effective potential (11) comes from the nonperturbative quantum contributions, and the factor $(2n)!/(2^n n!)$ of λ_{2n}^Q in Eqs. (12), (17), and (13) actually accounts for the number of symmetric loop diagrams from the higher order self-interactions and, hence, enhances the effective coupling constants in the very early Universe when the quantum effects of matter fields are expected to be important. For the massive real inflaton without a higher order interaction (i.e., $\lambda_{2n} = 0$), the wave function (7) determined in this way is indeed the exact state of Eq. (5), and the other excited quantum states can be constructed by acting on the creation operators (14) [13]. Once the higher order coupling is turned on, the Gaussian state is no longer exact, but it is also known that it describes quite accurately the system even for a strong coupling constant.

We are now able to find the two asymptotic solutions to Eq. (12). First, we consider the quantum dynamics at the very early Universe. Equation (12) can be written as a second order equation

$$\ddot{\chi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\chi} - \frac{1}{4a^6\chi^3} + m^2\chi + m_P^3\lambda_{2n}^Q\left(\frac{\chi}{m_P}\right)^{2n-1} = 0. \tag{20}$$

The symmetric Gaussian state (7) for the inflaton has the dispersions

$$\Delta\phi = \sqrt{\langle\hat{\phi}^2\rangle - \langle\hat{\phi}\rangle^2} = \chi,$$

$$\Delta\pi = \sqrt{\langle\hat{\pi}^2\rangle - \langle\hat{\pi}\rangle^2} = a^3(\dot{\chi}^2 + \chi^2\dot{\theta}^2)^{1/2}, \tag{21}$$

where $\langle\hat{\pi}\rangle = \langle\hat{\phi}\rangle = 0$. The uncertainty relation becomes

$$\Delta\phi\Delta\pi = \tfrac{1}{2}(1 + 4\pi_\chi^2\chi^2)^{1/2}, \tag{22}$$

where we used Eq. (18). A nearly minimal uncertainty can be achieved when $\pi_\chi\chi \sim 0$, that is, either $\pi_\chi \sim 0$ or $\chi \sim 0$. We propose an inflaton's initial quantum state with $\pi_\chi \sim 0$ and a large quantum fluctuation $\chi > \chi_c$, where

$$\frac{\chi_c}{m_P} = \min\left\{\left[\frac{1}{\lambda_{2n}^Q}\left(\frac{m}{m_P}\right)^2\right]^{1/(2n-2)}, \ \left[\frac{1}{4\lambda_{2n}^Q}\left(\frac{m}{a_0}\right)^6\right]^{1/(2n+2)}\right\}. \tag{23}$$

Under this initial condition the first term and the centrifugal potential term in Eq. (20) are small compared with the other two terms, and Eq. (20) reduces to the equation for classical inflaton when the dispersion χ is interpreted as ϕ. For this large quantum fluctuation, Eq. (20) may be approximated as

$$3\left(\frac{\dot{a}}{a}\right)\dot{\chi} + m_P^3\lambda_{2n}^Q\left(\frac{\chi}{m_P}\right)^{2n-1} \simeq 0. \tag{24}$$

Since the kinetic term $\pi_\chi^2/2a^3$ in $H_{\rm eff}$ is much smaller than $V_{\rm eff}$ for large χ, by substituting an approximate Friedmann equation

$$\frac{\dot{a}}{a} \simeq \sqrt{\frac{4\pi m_P^2\lambda_{2n}^Q}{3n}}\left(\frac{\chi}{m_P}\right)^n \tag{25}$$

into Eq. (24), we obtain the solutions

$$\chi(t) \simeq \chi(t_i)\exp\left(-\sqrt{\frac{m_P^2\lambda_{2n}^Q}{6\pi}}(t-t_i)\right) \quad (n=2), \tag{26}$$

$$\chi(t) \simeq \chi(t_i)\left[1 + \sqrt{\frac{n(n+2)^2 m_P^2\lambda_{2n}^Q}{12\pi}}(t-t_i)\right]^{1/(n-2)}, \quad (n \geqslant 3).$$

In all cases of n one has a period of inflation described by

$$a(t) \simeq a(t_i)\exp\left[\sqrt{\frac{4\pi m_P^2\lambda_{2n}^Q}{3n}}\int_{t_i}^{t}\left(\frac{\chi}{m_P}\right)^n(t)dt\right], \tag{27}$$

which follows from the Friedmann equation. These solutions are the same as those of the chaotic inflation model [2,6]. The big difference, however, is that in the quantum dynamical model of inflation the large quantum fluctuation with a nearly minimal uncertainty and the nonperturbative quantum contributions at the very early Universe do drive the quasi-exponential expansion of the Universe. As the Universe inflates, $\Delta\phi$ decreases but $\Delta\pi_\chi$ grows and the symmetric Gaussian state becomes sharply peaked, showing classical features.

Second, we consider the late evolution of the Universe. Using nonlinear system theory [14], we find a *limit cycle* when $(\partial/\partial\chi)\, V_{eff}(\chi) = 0$:

$$\frac{1}{4a^6\chi^4} = m^2 + m_P^2\lambda_{2n}^Q\left(\frac{\chi}{m_P}\right)^{2(n-1)}. \tag{28}$$

As the Universe expands, the quantum fluctuation χ decreases due to the friction from the Hubble parameter and behaves as a classical inflaton field. The equilibrium is determined dominantly by the mass term, and the motion of the

[2]For a systematic improvement of the approximation, see Ref. [9], and the detailed deviations from the exact results in this Gaussian approximation are dealt with in Ref. [11].

effective inflaton tends toward the limit cycle of a coherent oscillation constrained approximately by

$$\chi^2 a^3 \simeq \frac{1}{2m}. \tag{29}$$

The Friedmann equation along the limit cycle of the motion leads to a power-law expansion

$$a(t) \simeq \left[\frac{3\pi m}{2m_P^2} \left(1 + \frac{1}{n} \right) t^2 \right]^{1/3}, \tag{30}$$

which is valid for the later time.

Finally, we extend the potential in Eq. (2) to be an arbitrary analytic and bound potential $V(\phi) = F(\phi)$, which is also symmetric, $F(-x) = F(x)$. One may then expand the potential in a Taylor series by

$$F(\phi) = \sum_{n=1} \frac{m_P^4 \lambda_{2n}}{(2n)!} \left(\frac{\phi}{m_P} \right)^{2n}, \tag{31}$$

where the vacuum energy is adjusted to zero. By the same analysis as before, one may obtain the effective potential from quantum fluctuation

$$V_{\mathrm{eff}} = \frac{1}{8a^6\chi^2} + \sum_{n=1} \frac{m_P^4 \lambda_{2n}}{2^n n!} \left(\frac{\chi^2}{m_P^2} \right)^n. \tag{32}$$

Now, most of the above analysis of the nonlinear system holds with a modification that the lowest term n_{min} plays the role of determining the limit cycle and the largest term n_{max} is driving an inflation.

In summary, we have studied the quantum dynamics of a real inflaton in its symmetric Gaussian state for a large class of potentials. We have found an equivalent two-dimensional nonlinear dynamical system for the dispersion of the inflaton and the conserved U(1) charge as an angular momentum. We have put forward the minimal uncertainty proposal that the initial condition with a large field fluctuation (dispersion) and a small momentum and with an overall minimal uncertainty naturally explains the early stage of the inflationary universe. It was shown that the U(1) charge is fixed to a definite value, which plays the role of angular momentum in the two-dimensional nonlinear system. This is contrasted to the usual complex inflaton theory used in the inflation model [15,16] or the wormhole model [17] since there is no *a priori* reason to take the U(1) charge to be fixed in these models.

We conclude with some comments that we have not dealt in detail with the enhancement of effective coupling constants and with the inhomogeneous degrees of freedom (i.e., the nonzero modes) in the real scalar field. Analysis of the nonzero modes may be interesting since they give rise to the density perturbation necessary for structure formation and may affect the inflation and the quantum dynamics of the real inflaton. The quantization of this interacting theory of the nonzero modes and the role of effective coupling constants in density perturbation require a further study.

This work was supported in parts by Center for Theoretical Physics, Seoul National University, and by the Basic Science Research Institute Program, Korea Ministry of Education, under Projects Nos. BSRI-97-2418, BSRI-97-2425, and BSRI-97-2427. S.P.K. was also supported by the Non Directed Research Fund, Korea Research Foundation, and J.H.Y. by KOSEF Grant No. 97-07-02-02-01-3.

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